Intertwining Relations for the Deformed D1D5 CFT

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Abstract

The Higgs branch of the D1D5 system flows in the infrared to a two-dimensional $\mathcal{N} = (4, 4)$ SCFT. This system is believed to have an "orbifold point" in its moduli space where the SCFT is a free sigma model with target space the symmetric product of copies of T^4; however, at the orbifold point gravity is strongly coupled and to reach the supergravity point one needs to turn on the four exactly marginal deformations corresponding to the blow-up modes of the orbifold SCFT. Recently, technology has been developed for studying these deformations and perturbing the D1D5 CFT off its orbifold point. We present a new method for computing the general effect of a single application of the deformation operators. The method takes the form of intertwining relations that map operators in the untwisted sector before application of the deformation operator to operators in the 2-twisted sector after the application of the deformation operator. This method is computationally more direct, and may be of theoretical interest. This line of inquiry should ultimately have relevance for black hole physics.

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1. INTRODUCTION

The D1D5 system has been a very rich area for exploring black hole physics: one can account for the black hole entropy by counting states of the dual CFT [1–3]; one can compute the spectrum and rate of absorption and emission from the gravitational description and match it to a CFT description [4–16]; and one can create and study explicit microstate geometries (see [17–21] for reviews of older work, and [22–25] for some recent developments).

There is believed to be an orbifold point in the D1D5 moduli space where a free two-dimensional CFT with orbifolded target space becomes a good description [26–32]. Because calculations are easier at the orbifold point, all of the above mentioned CFT calculations are done there. At the orbifold point, however, the gravity approximation is not valid. While the entropy-counting calculations are protected on account of supersymmetry, the agreement of so many absorption and emission calculations at the orbifold point with supergravity calculations at a different point in moduli space strongly suggests some kind of "non-renormalization" theorem [33].

To understand more complicated black hole physics, it becomes necessary to deform the CFT off of the orbifold point by blowing up the fixed points of the orbifold. We do this by introducing four marginal deformation operators.[1] While the supergravity point in moduli space is far from the orbifold point, we take a perturbative approach and start by looking at the action of a single application of the deformation operator. We hope that this approach will yield insight into black hole physics. To this end, [34] computes the effect of the deformation operator on the vacuum, and [35] shows an algorithm for computing the effect of the deformation operator on more general states of the CFT.

In this paper, we introduce a new and more direct method for computing the effect of the deformation operator on general states. This method may also be of theoretical interest, because it makes the physics of twist operators closer to that of Bogolyubov transformations. We start by looking for a Bogolyubov-type relation between operators "before" the deformation and "after" the deformation. Unfortunately, the method suffers from multidimensional series whose value depends sensitively on the way one evaluates them. We resolve these ambiguities by introducing a physically motivated prescription for evaluating the infinite series; however, the prescription makes our relation a weaker statement, since the prescription depends on what other excitations one has. Thus, we term the connection between operators before the deformation and after the deformation "intertwining relations" instead of Bogolyubov coefficients.

In Section 2, we briefly introduce the deformation operator for the D1D5 CFT and review results of [34]. In particular, the deformation operator contains a 2-twist operator, whose effect on the vacuum is to produce a "squeezed state." The squeezed state, written as an exponential of pairs of bosonic and fermionic modes acting on the twisted vacuum, is reminiscent of the state one generically gets from a Bogolyubov transformation of the vacuum. The twist operator is the only difficult part of working with the deformation operator, and for this paper we restrict our attention to it.

In Section 3, we make the above analogy more precise by introducing the analogue of the Bogolyubov coefficients, that is we show how the 2-twist operator maps modes of the untwisted sector into twisted sector modes. We first give a simple, formal derivation of these intertwining coefficients. Then we give an example of the order-of-summation issue that results from using the coefficients. We can understand the problem by making a more rigorous derivation of the intertwining relations, which allows us to give a prescription that gives the correct answer. In 3.3.4

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[1] The four deformation operators form a 2-index tensor of $SU(2)_2$, so we sometimes call this object "the deformation operator."

we concisely state the prescription.

In Section 4, we demonstrate the prescription and the validity of the more rigorous derivation by computing a particular state in two distinct ways and showing agreement.

Finally, we recapitulate the main points and discuss future directions in the conclusion. Our notation and conventions for the CFT are outlined in Appendix A. Appendix B contains a list of series that we use. Some of them we could not prove directly, but we are confident are correct after numerical comparisons.

2. THE DEFORMATION OPERATOR

Recall that the D1D5 system is IIB string theory compactified on[2]

$$M_{9,1} \to M_{4,1} \times S^1 \times T^4, \tag{2.1}$$

with a bound state of N_1 D1 branes wrapping S^1 and N_5 D5 branes wrapping $T^4 \times S^1$. For S^1 large compared to the T^4, the low energy description of the resulting bound state is a 1+1-dimensional CFT. The CFT has $SU(2)_L \times SU(2)_R$ R-symmetry that corresponds to the isometry of the near-horizon S^3. Additionally, we find it convenient to label fields with the $SO(4)_I \simeq SU(2)_1 \times SU(2)_2$ symmetry corresponding to the torus directions [32]. This symmetry is broken by the torus, but is still useful to organize the fields.

It has been conjectured that we can move to a point in the moduli space of the D1D5 system called the "orbifold point," where the CFT is a 1+1-dimensional sigma model with target space $(T^4)^{N_1 N_5}/S_{N_1 N_5}$ [26–32]. Each of the $N_1 N_5$ copies consist of 4 real bosonic fields X and four real fermionic fields ψ, see Appendix A and [34, 35] for more details. We parameterize the base space with one complex coordinate z, using radial quantization.

The orbifold theory has different twist sectors, which correspond to circling S^1 and coming back only up an element of the orbifold group, $S_{N_1 N_5}$. The twist operators σ_n map between different twist sectors. We do not need most of the technical details of the twist operators in this paper, but we do use the fact that one can map to a covering space which looks like the untwisted sector of the theory via a meromorphic mapping $z = z(t)$ [36].

For this paper, we only need the chiral primary 2-twist operator $\sigma_2^{++}(z)$, where the plusses indicate that it is the top member of both a $SU(2)_L$ doublet and a $SU(2)_R$ doublet. If the only twist operators are a $\sigma_2^{++}(z_0)$ and another 2-twist at $z = \infty$, then

$$z = z_0 + t^2 \tag{2.2}$$

maps to a single-valued covering space parameterized by the complex coordinate t. This map removes the twisting, but the $SU(2)_L \times SU(2)_R$ charge is preserved by inserting spin fields, $S^+(t = 0)\bar{S}^+(\bar{t} = 0)$, which also ensure the correct fermion boundary conditions [37].

The D1D5 CFT has 20 exactly marginal deformation operators [33, 38–40] that correspond to different directions in moduli space. Four of those operators are the blow-up modes of the orbifold. These are the deformation operators that move us to the supergravity point in the moduli space and hence the ones which we are most interested in.

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[2] One could consider also K3 instead of T^4; however, for simplicity we consider only T^4.

2.1. The operator

The deformation operator is a singlet under $SU(2)_L \times SU(2)_R$. To obtain such a singlet we apply modes of $G_{\dot{A}}^{\mp}$ to $\sigma_2^{\pm}$. In [34] it was shown that we can write the deformation operator(s) as

$$\widehat{O}_{\dot{A}\dot{B}}(w_0) = \left[\int_{w_0} \frac{dw}{2\pi i} G_{\dot{A}}^{-}(w) \right] \left[\int_{\bar{w}_0} \frac{d\bar{w}}{2\pi i} \bar{G}_{\dot{B}}^{-}(\bar{w}) \right] \sigma_2^{++}(w_0) \tag{2.3}$$

The left and right movers separate out completely for all the computations that we perform. Thus from now on we work with the left movers only; in particular, we write the twist operator only with its left spin: σ_2^{+}.

The operator σ_2^{+} is normalized to have a unit OPE with its conjugate

$$\sigma_{2,+}(z')\sigma_2^{+}(z) \sim \frac{1}{(z'-z)}. \tag{2.4}$$

This implies that acting on the Ramond vacuum [34]

$$\sigma_2^{+}(z)|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = |0_R^-\rangle + O(z). \tag{2.5}$$

Here $|0_R^-\rangle$ is the spin down Ramond vacuum of the CFT on the doubly wound circle produced after the twist. The normalization (2.5) has given us the coefficient unity for the first term on the RHS, and the $O(z)$ represent excited states of the CFT on the doubly wound circle.

In this paper, we focus on the effect of the twist operator since the effect of the susy-current, G, is easy to compute: one can break the G-contours in Equation (2.3) into contours acting before the twist operator and contours acting after the twist operator [34, 35].

2.2. The action of the twist operator on the vacuum

Since we are mainly interested in better understanding the twist operator, it is useful to define

$$|\chi\rangle = \sigma_2^{+}(z_0)|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)}, \tag{2.6}$$

and recall that in [34] it was found that the twist operator acting on the Ramond vacuum may be written as the squeezed state

$$|\chi\rangle = \exp\left[-\frac{1}{2} \sum_{m,n} \gamma_{mn}^B \epsilon^{AB} \epsilon^{\dot{A}\dot{B}} \alpha_{A\dot{A},-m} \alpha_{B\dot{B},-n} + \sum_{m,n} \gamma_{mn}^F \epsilon_{AB} \psi_{-m}^{+A} \psi_{-n}^{-B} \right] |0_R^-\rangle \tag{2.7}$$

where

$$\gamma_{mn}^B = \begin{cases} \frac{4 z_0^{\frac{m+n}{2}}}{mn(m+n)\pi} \frac{\Gamma(\frac{m}{2}+1)\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})} & m, n \text{ odd, positive} \\ 0 & \text{otherwise} \end{cases} \tag{2.8a}$$

$$\gamma_{mn}^F = \begin{cases} -\frac{2 z_0^{\frac{m+n}{2}}}{n(m+n)\pi} \frac{\Gamma(\frac{m}{2}+1)\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})} & m, n \text{ odd, positive} \\ 0 & \text{otherwise.} \end{cases} \tag{2.8b}$$

3. INTERTWINING RELATIONS

We first present a formal, intuitive derivation of the intertwining relations, which relate modes of the untwisted sector to modes of the twisted sector. In Section 3.3, we show how the derived relations lead to multi-dimensional infinite series whose value depends on how one takes the limit of the partial sums going to infinity. We then give a more rigorous derivation of the relations, which suggests a prescription on how to evaluate the ambiguous series.

3.1. Basic Derivation

We are interested in finding a relationship between modes in the untwisted sector, "before the twist," and modes in the twisted sector, "after the twist." More precisely we would like to know in general how to write

$$\sigma_2^+(z_0)\big(\text{excitations before the twist}\big)|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = \big(\text{excitations after the twist}\big)\sigma_2^+(z_0)|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)}$$
$$= \big(\text{excitations after the twist}\big)|\chi\rangle. \tag{3.1}$$

In [35], an algorithm was found for doing just that; however, the question this paper addresses is whether there is a more general relation between individual modes before the twist operator and after the twist operator.

Since our argument does not depend on $SU(2)_L$ or $SU(2)_1 \times SU(2)_2$, we suppress the indices on the bosons and fermions. We begin by noting that before the twist the correct field expansions are

$$i\partial X^{(1)}(z) = \sum_n \frac{\alpha_n^{(1)}}{z^{n+1}} \qquad i\partial X^{(2)}(z) = \sum_n \frac{\alpha_n^{(2)}}{z^{n+1}} \qquad |z| < |z_0|, \tag{3.2}$$

and

$$\psi^{(1)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\psi_n^{(1)}}{z^{n+\frac{1}{2}}} \qquad \psi^{(2)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\psi_n^{(2)}}{z^{n+\frac{1}{2}}} \qquad |z| < |z_0|. \tag{3.3}$$

After the twist, the correct expansions are given by

$$i\partial X^{(1)}(z) = \frac{1}{2}\sum_n \frac{\alpha_n}{z^{\frac{n}{2}+1}} \qquad i\partial X^{(2)}(z) = \frac{1}{2}\sum_n \frac{(-1)^n\alpha_n}{z^{\frac{n}{2}+1}} \qquad |z| > |z_0|, \tag{3.4}$$

and

$$\psi^{(1)}(z) = \frac{1}{2}\sum_n \frac{\psi_n}{z^{\frac{n}{2}+\frac{1}{2}}} \qquad \psi^{(2)}(z) = \frac{1}{2}\sum_n \frac{(-1)^n\psi_n}{z^{\frac{n}{2}+\frac{1}{2}}} \qquad |z| > |z_0|. \tag{3.5}$$

Note that after the twist, there is no unique way of distinguishing copies (1) and (2), but the above expansions correspond to *a* way of defining (1) and (2). Following [34, 35], we define the modes in the twisted sector with an extra factor of 2 so that we can work with integers.

Now, recall that the fields $\partial X(z)$ and $\psi(z)$ should be holomorphic functions except at isolated points where there are other operator insertions. In particular, there is nothing special that occurs on the circle $|z| = |z_0|$ (there *is* something special at the isolated point z_0). The curve is the boundary between our twisted and untwisted mode expansions about the origin $z = 0$, but if one were to do mode expansions about a different point in the complex plane then the expansions would

change across a different curve (that still passes through z_0). Therefore, *at least away from the twist operator at z_0, we expect that the fields $\partial X(z)$ and $\psi(z)$ should be continuous across $|z| = |z_0|$.*

Thus, on the circle $|z| = |z_0|$ (*excluding some neighborhood around $z = z_0$*) we may identify

$$\sum_n \frac{\alpha_n^{(1)}}{z^{n+1}} = \frac{1}{2} \sum_n \frac{\alpha_n}{z^{\frac{n}{2}+1}} \qquad \sum_n \frac{\alpha_n^{(2)}}{z^{n+1}} = \frac{1}{2} \sum_n \frac{(-1)^n \alpha_n}{z^{\frac{n}{2}+1}}, \qquad |z| = |z_0| \tag{3.6}$$

and

$$\sum_{n \in \mathbb{Z}+\frac{1}{2}} \frac{\psi_n^{(1)}}{z^{n+\frac{1}{2}}} = \frac{1}{2} \sum_n \frac{\psi_n}{z^{\frac{n}{2}+\frac{1}{2}}} \qquad \sum_{n \in \mathbb{Z}+\frac{1}{2}} \frac{\psi_n^{(2)}}{z^{n+\frac{1}{2}}} = \frac{1}{2} \sum_n \frac{(-1)^n \psi_n}{z^{\frac{n}{2}+\frac{1}{2}}}, \qquad |z| = |z_0|. \tag{3.7}$$

Multiplying (3.6) by z^m and integrating along the circle $|z| = |z_0|$, we get

$$\begin{aligned}
\alpha_m^{(1)} &= \frac{1}{2} \sum_n \alpha_n \int \frac{dz}{2\pi i} z^{m - \frac{n}{2} - 1} \\
&= \frac{1}{2} \sum_n \alpha_n \int_0^{2\pi} \frac{d\theta}{2\pi} (z_0 e^{i\theta})^{m - \frac{n}{2}} \\
&= \frac{1}{2} \alpha_{2m} + \frac{i}{2\pi} \sum_{n \text{ odd}} \frac{z_0^{m - \frac{n}{2}}}{m - \frac{n}{2}} \alpha_n,
\end{aligned} \tag{3.8}$$

and similarly

$$\alpha_m^{(2)} = \frac{1}{2} \alpha_{2m} - \frac{i}{2\pi} \sum_{n \text{ odd}} \frac{z_0^{m - \frac{n}{2}}}{m - \frac{n}{2}} \alpha_n, \tag{3.9}$$

where the sum over the odds is both positive and negative. These are the desired relations between modes before the twist and modes after the twist. The relations are analogous to the more general Bogolyubov transformations discussed in condensed matter in [41]. Note that the contour in (3.8) is open, since we must exclude some infinitesimal neighborhood around z_0. It is straightforward to find the analogous relation for fermions in the R sector:

$$\psi_n^{(1,2)} = \frac{1}{2} \psi_{2n} \pm \frac{i}{2\pi} \sum_{k \text{ odd}} \frac{z_0^{n - \frac{k}{2}}}{n - \frac{k}{2}} \psi_k. \tag{3.10}$$

Given the delicate nature of the above argument, in particular with regard to what is happening around z_0, one may not be surprised that there are some hidden subtleties with these relations. Note also that the derivation would seem to work for any holomorphic field $\mathcal{O}(z)$, which *cannot be correct* since some modes have a nontrivial commutator with the twist operator, e.g. J_0^-. We should be careful in what we mean when we write "$=$" in the above expressions. For example in Equation (3.8), we implicitly mean the operator relation

$$\sigma_2^+(z_0) \alpha_m^{(1)} = \left[\frac{1}{2} \alpha_{2m} + \frac{i}{2\pi} \sum_{n \text{ odd}} \frac{z_0^{m - \frac{n}{2}}}{m - \frac{n}{2}} \alpha_n \right] \sigma_2^+(z_0) \tag{3.11}$$

with the above radial ordering. The usage should be clear from the context.

Before showing what goes wrong, let us first explore what goes right. First of all this is the kind of relation we were hoping for: it relates positive and negative modes before the twist to

positive and negative modes after the twist directly. The method given in [35] can only relate states to states; it cannot relate an individual mode before the twist to modes after the twist without knowing what other excitations one has before the twist.

An important requirement for Bogolyubov coefficients is that they respect the commutation relations. We can check that the above relations are consistent with the commutation relations by computing, for instance,

$$
\begin{aligned}
\left[\alpha_m^{(1)}, \, \alpha_n^{(1)}\right] &= \left[\frac{1}{2}\alpha_{2m} + \frac{i}{2\pi}\sum_{k \text{ odd}} \frac{z_0^{m-\frac{k}{2}}}{m - \frac{k}{2}}\alpha_k, \, \frac{1}{2}\alpha_{2n} + \frac{i}{2\pi}\sum_{l \text{ odd}} \frac{z_0^{n-\frac{l}{2}}}{n - \frac{l}{2}}\alpha_l\right] \\
&= \frac{m}{2}\delta_{m+n,0} - \frac{z_0^{m+n}}{4\pi^2}\sum_{k \text{ odd}} \frac{k}{(m - \frac{k}{2})(n + \frac{k}{2})}.
\end{aligned}
\tag{3.12}
$$

The sum is divergent; however, if we make the relatively modest assumption that we should cutoff the sum symmetrically for positive and negative k, then we find[3]

$$
\lim_{L\to\infty} \sum_{k \text{ odd}}^{|k|<L} \frac{k}{(m - \frac{k}{2})(n + \frac{k}{2})} = -2m\pi^2\delta_{m+n,0},
\tag{3.13}
$$

which gives the correct answer. Similar calculations go through for the other (anti-)commutations.

Second, it gets the right answer for moving a single mode through the twist operator. For instance,

$$
\begin{aligned}
\sigma_2^+(z_0)\alpha_{A\dot{A},n}^{(1)}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} &= \left(\frac{1}{2}\alpha_{A\dot{A},2n} + \frac{i}{2\pi}\sum_{k \text{ odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}}\alpha_{A\dot{A},k}\right)\sigma_2^+(z_0)|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} \\
&= \left(\frac{1}{2}\alpha_{A\dot{A},2n} + \frac{i}{2\pi}\sum_{k \text{ odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}}\alpha_{A\dot{A},k}\right)|\chi\rangle \\
&= \left[\frac{1}{2}\alpha_{A\dot{A},2n} + \frac{i}{2\pi}\left(\sum_{k \text{ odd}^+} \frac{z_0^{n+\frac{k}{2}}}{n + \frac{k}{2}}\alpha_{A\dot{A},-k} + \sum_{k,l \text{ odd}^+} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}}\gamma_{kl}^B k\alpha_{A\dot{A},-l}\right)\right]|\chi\rangle.
\end{aligned}
\tag{3.14}
$$

Making use of the identity in (B.1) one sees that

$$
\sum_{k \text{ odd}^+} \frac{z_0^{n-\frac{k}{2}}k\gamma_{kl}^B}{n - \frac{k}{2}} = \frac{z_0^{n+\frac{l}{2}}}{n + \frac{l}{2}}\left(\frac{\Gamma(\frac{l}{2})\Gamma(-n + \frac{1}{2})}{\Gamma(\frac{l+1}{2})\Gamma(-n)} - 1\right),
\tag{3.15}
$$

and thus

$$
\sigma_2^+(z_0)\alpha_{A\dot{A},n}^{(1)}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = \left[\frac{1}{2}\alpha_{A\dot{A},2n} + \frac{i}{2\pi}\sum_{l \text{ odd}^+} \frac{z_0^{n+\frac{l}{2}}}{n + \frac{l}{2}}\frac{\Gamma(\frac{l}{2})\Gamma(-n + \frac{1}{2})}{\Gamma(\frac{l+1}{2})\Gamma(-n)}\alpha_{A\dot{A},-l}\right]|\chi\rangle.
\tag{3.16}
$$

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[3] This mild UV ambiguity might be seen as a hint of the other UV issues with the intertwining relations; however, this issue does not arise for the fermions, and it is of a different character. The UV issues that we discuss at length arise with multidimensional series; whereas the above is arguably the only reasonable regularization of the series in (3.12). For example, if one cuts off the positive modes at L and the negative modes at $-2L$ then one *does* get a different answer, but it is not a consistent truncation of the Hilbert space to have a creation operator without its corresponding annihilation operator.

For n positive the above vanishes as it should, since the positive even mode in the first term annihilates $|\chi\rangle$ and $\Gamma(-n)$ kills the second term. For n negative this reproduces the result found in [35] for a single mode in the initial state.

Similarly, if one performs the analogous calculation for fermions with (3.10), then one can use

$$\psi_{+k}^{\alpha A}|\chi\rangle = 2\sum_{p\,\mathrm{odd}^+}\left(\gamma_{pk}^F\delta_+^\alpha\psi_{-p}^{+A} - \gamma_{kp}^F\delta_-^\alpha\psi_{-p}^{-A}\right)|\chi\rangle \qquad k\text{ odd, positive,} \tag{3.17}$$

to find

$$\sigma_2^+(z_0)\psi_n^{(1)+A}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = \left[\frac{1}{2}\psi_{2n}^{+A} + \frac{i}{2\pi}\sum_{p\,\mathrm{odd}^+}\frac{z_0^{n+\frac{p}{2}}}{n+\frac{p}{2}}\frac{\Gamma(\frac{p}{2}+1)\Gamma(-n+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n+1)}\psi_{-p}^{+A}\right]|\chi\rangle \tag{3.18a}$$

$$\sigma_2^+(z_0)\psi_n^{(1)-A}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = \left[\frac{1}{2}\psi_{2n}^{-A} + \frac{i}{2\pi}\sum_{p\,\mathrm{odd}^+}\frac{z_0^{n+\frac{p}{2}}}{n+\frac{p}{2}}\frac{\Gamma(\frac{p}{2})\Gamma(-n+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n)}\psi_{-p}^{-A}\right]|\chi\rangle. \tag{3.18b}$$

This agrees with [35].

3.2. Problems

There are two problems with the above derivation. One is that this formal derivation only makes use of the holomorphicity of the fields, which means that one could make the same argument for any other holomorphic field. For instance, consider $J^a(z)$, one would get

$$J_n^{a(1,2)} \overset{?}{=} \frac{1}{2}J_{2n}^a \pm \frac{i}{2\pi}\sum_{k\,\mathrm{odd}}\frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}}J_k^a, \tag{3.19}$$

which leads to

$$\left[J_0^-, \sigma_2^+(z_0)\right] = J_0^-\sigma_2^+(z_0) - \sigma_2^+(z_0)\left(J_0^{(1)} + J_0^{(2)}\right)$$
$$\overset{?}{=} 0. \tag{3.20}$$

One should find $\sigma_2^-(z_0)$, not zero. One finds similar contradictions if one tries to use the same argument for $T(z)$, too.

The second problem, alluded to above, concerns using the intertwining relations with more than one mode. The simplest instance may be to consider

$$\sigma_2^+(z_0)\alpha_{++,m}^{(1)}\alpha_{--,-n}^{(1)}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)} = -m\delta_{m,n}|\chi\rangle \qquad m,n > 0. \tag{3.21}$$

If we use Equation (3.8), then we get

$$-m\delta_{m,n}|\chi\rangle \overset{?}{=} \left(\frac{1}{2}\alpha_{++,2m} + \frac{i}{2\pi}\sum_{k\,\mathrm{odd}}\frac{z_0^{m-\frac{k}{2}}}{m-\frac{k}{2}}\alpha_{++,k}\right)\left(\frac{1}{2}\alpha_{--,-2n} + \frac{i}{2\pi}\sum_{l\,\mathrm{odd}}\frac{z_0^{-n-\frac{l}{2}}}{-n-\frac{l}{2}}\alpha_{--,l}\right)|\chi\rangle$$

$$= -\frac{m}{2}\delta_{m,n}|\chi\rangle - \frac{1}{4\pi^2}\left(\sum_{k\,\mathrm{odd}}\frac{z_0^{m-\frac{k}{2}}}{m-\frac{k}{2}}\alpha_{++,k}\right)\left(\sum_{l\,\mathrm{odd}}\frac{z_0^{-n-\frac{l}{2}}}{-n-\frac{l}{2}}\alpha_{--,l}\right)|\chi\rangle$$

$$= -\frac{m}{2}\delta_{m,n}|\chi\rangle - \frac{1}{4\pi^2}\left(\sum_{k\,\mathrm{odd}}\frac{z_0^{m-\frac{k}{2}}}{m-\frac{k}{2}}\alpha_{++,k}\right)\left[\sum_{j\,\mathrm{odd}^+}\alpha_{--,-j}\left(\frac{z_0^{-n+\frac{j}{2}}}{-n+\frac{j}{2}} + \sum_{l\,\mathrm{odd}^+}\frac{z_0^{-n-\frac{l}{2}}l\gamma_{lj}}{-n-\frac{l}{2}}\right)\right]|\chi\rangle \tag{3.22}$$

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Note that the even–odd cross-terms vanish since they commute and either the $\alpha_{++,2m}$ or the $\alpha_{++,k}$-sum kills $|\chi\rangle$ (from (3.16)). Similarly, we need only look at the commutator of the $\alpha_{++,k}$-sum and the square-bracketed expression, which gives

$$
-\left(\sum_{k\,\text{odd}^+} \frac{k z_0^{m-\frac{k}{2}}}{m-\frac{k}{2}}\right)\left(\frac{z_0^{-n+\frac{k}{2}}}{-n+\frac{k}{2}} + \sum_{l\,\text{odd}^+} \frac{z_0^{-n-\frac{l}{2}} l \gamma_{lk}}{-n-\frac{l}{2}}\right)
$$
$$
= z_0^{m-n} \sum_{k\,\text{odd}^+}\left(\frac{k}{(m-\frac{k}{2})(n-\frac{k}{2})} + \sum_{l\,\text{odd}^+} \frac{z_0^{-\frac{k}{2}-\frac{l}{2}} kl \gamma_{kl}}{(m-\frac{k}{2})(n+\frac{l}{2})}\right). \tag{3.23}
$$

We would like the above expression to evaluate to $2\pi^2 m \delta_{m,n}$ in order to get the correct answer; however, the above summations depend sensitively on the order in which one adds the infinite number of terms. For instance, if we attempt to perform the k-sum *first*, then the first term is divergent and the second term, using (3.15), gives

$$
-\sum_{l\,\text{odd}^+} \frac{l}{(m+\frac{l}{2})(n+\frac{l}{2})}, \tag{3.24}
$$

which is also divergent.

On the other hand, if we perform the l sum first, using (3.15) we find

$$
z_0^{m-n} \sum_{k\,\text{odd}^+}\left(\frac{k}{(m-\frac{k}{2})(n-\frac{k}{2})} + \sum_{l\,\text{odd}^+} \frac{z_0^{-\frac{k}{2}-\frac{l}{2}} kl \gamma_{kl}}{(m-\frac{k}{2})(n+\frac{l}{2})}\right)
$$
$$
= z_0^{m-n} \sum_{k\,\text{odd}^+}\left(\frac{k}{(m-\frac{k}{2})(n-\frac{k}{2})} + \frac{k}{(m-\frac{k}{2})(n-\frac{k}{2})}\left[\frac{\Gamma(\frac{k}{2})\Gamma(n+\frac{1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(n)} - 1\right]\right)
$$
$$
= z_0^{m-n} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} \sum_{k\,\text{odd}^+}\left(\frac{k}{(m-\frac{k}{2})(n-\frac{k}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})}\right), \tag{3.25}
$$

which using another identity,

$$
\sum_{k\,\text{odd}^+} \frac{k}{(m-\frac{k}{2})(n-\frac{k}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})} = 2\pi^2 m \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} \delta_{m,n} \qquad m, n > 0, \tag{3.26}
$$

gives the correct answer.

How should we think of the ambiguity in Equation (3.23)? Any infinite series is implicitly evaluated by determining the limit of a sequence of *partial sums*. In our case, higher values of k and l correspond to higher modes, so it is natural to think of imposing UV cutoffs on the sums, $k < L_1$ and $l < L_2$. We then wish to take the limit as $L_1, L_2 \to \infty$, but there are many different ways to do that. If we define $b = L_1/L_2$ to parameterize the different ways of evaluating (3.23), then evaluating the k-sum first corresponds to $b = \infty$, while evaluating the l-sum first corresponds to $b = 0$. These are just two of an infinite number of ways to evaluate the double-sum.

Given that these ambiguous multi-dimensional series are rampant in this formalism and that frequently the correct method of evaluating them may be much less obvious,[4] we need a well-motivated principle that determines the correct way to handle the UV physics.

---

[4] For instance, there are cases involving triple-sums, where more than one order of evaluating the sums give distinct, finite results.

3.3. A More Rigorous Derivation

We now present a more rigorous derivation of the intertwining relations in Equations (3.8), (3.9), and (3.10). By continuously deforming the contour integral for an initial state mode outward only where the integrand is holomorphic, we can treat the point z_0 more carefully. This resolves the two problems outlined above.
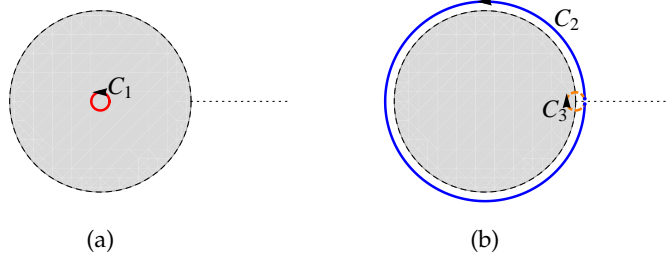

(a) (b)

FIG. 1: In the z-plane, showing how the contour C_1 (solid, red) in (a) may be deformed out and around the branch cut into contours C_2 (solid, blue) and C_3 (dashed, orange) in (b). The gray circular region is the "before the twist" region, $|z| < |z_0|$. The branch cut is indicated by the dashed black line extending out from the circle.

3.3.1. Bosons

Working with the bosons first, let us note that

$$\alpha_n^{(1)} = \oint_{C_1} \frac{dz}{2\pi i} i\partial X^{(1)}(z) z^n, \tag{3.27}$$

where C_1 is a circular contour with radius less than $|z_0|$ shown in Figure 1(a), and $i\partial X^{(1)}(z)$ is a holomorphic function except at $z = z_0$ (and excluding any other operator insertions). Thus, we may deform the contour into an open circle C_2 of radius larger than $|z_0|$ and a contour, C_3, sneaking around the branch cut starting at $z = z_0$, as shown in Figures 1 and 2. We take C_3 to be a circle of radius ε, which we eventually take to zero. The orientations of the contours are shown in the figures.

We write the above contour integral as

$$\alpha_n^{(1)} = \int_{C_2} \frac{dz}{2\pi i} i\partial X^{(1)}(z) z^n + \int_{C_3} \frac{dz}{2\pi i} i\partial X^{(1)}(z) z^n. \tag{3.28}$$
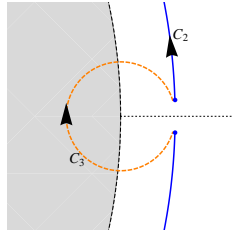

FIG. 2: A close-up depiction of contours C_2 and C_3 meeting. Note that we have added an artifical gap around the branch cut for illustrative purposes only—in fact, both C_2 and C_3 are full circles.
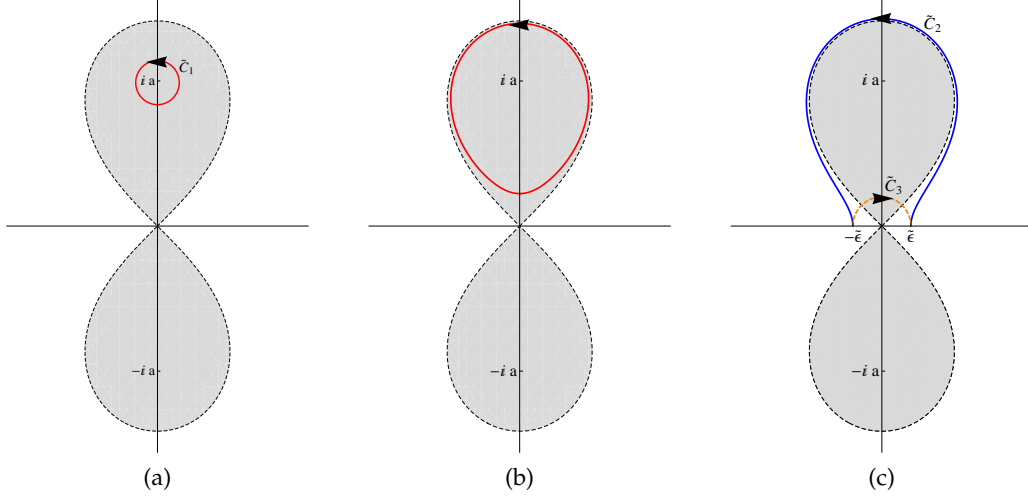
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FIG. 3: We show the t-plane where there is no branch cut and the fields are single-valued. We start in (a) with the $\tilde{C}_1$ contour (solid, red), which we finally deform out into $\tilde{C}_2$ (solid, blue) and $\tilde{C}_3$ (dashed, orange) in (c).

The C_2 term, is what we have been calculating and is given by (with ε corrections)

$$
\int_{C_2} \frac{\mathrm{d}z}{2\pi i} i\partial X(z) z^n = \frac{i}{2} \sum_k \alpha_k \int_{C_2} \frac{\mathrm{d}z}{2\pi i} z^{n-\frac{k}{2}-1}
$$

$$
= \frac{1}{4\pi} \sum_k \alpha_k (z_0+\varepsilon)^{n-\frac{k}{2}} \int_0^{2\pi} \mathrm{d}\theta\, e^{i(n-\frac{k}{2})\theta}
$$

$$
= \frac{1}{2}\alpha_{2n} + \frac{i}{2\pi} \sum_{k\ \mathrm{odd}} \frac{(z_0+\varepsilon)^{n-\frac{k}{2}}}{n-\frac{k}{2}} \alpha_k. \tag{3.29}
$$

At this point, it becomes necessary to introduce the covering space where the fields are well-defined. We map to the covering space coordinate t via

$$
z = z_0 + t^2 \qquad a^2 = z_0. \tag{3.30}
$$

The points ia and $-ia$ are the two images of the origin $z=0$, one corresponding to each copy of the fields. Mapping the z-plane contours in Figure 1 to the t-plane results in Figure 3.

We compute the C_3 term by going to the t-plane

$$
\int_{C_3} \frac{\mathrm{d}z}{2\pi i} \partial X(z) z^n = \int_{\tilde{C}_3} \frac{\mathrm{d}t}{2\pi i} \partial X(t)(z_0+t^2)^n \xrightarrow[\tilde{\varepsilon}\to 0]{} 0, \tag{3.31}
$$

where the radius of the semicircular contour in the t-plane, $\tilde{C}_3$, is given by $\tilde{\varepsilon}$. The contour gives zero contribution since there is no bosonic insertion at $t=0$ and the integrand is therefore analytic. As the length of the contour goes to zero, therefore, so does the integral. Thus, we see that the story for the bosons is exactly as stated, and in the limit as $\varepsilon \to 0$ we reproduce our previous result,

$$
\alpha_n^{(1,2)} = \frac{1}{2}\alpha_{2n} \pm \frac{i}{2\pi} \sum_{k\ \mathrm{odd}} \frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}} \alpha_k. \tag{3.32}
$$

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3.3.2. Fermions

For the fermions we have

$$\psi_n^{(1)} = \oint_{C_1} \frac{\mathrm{d}z}{2\pi i} \psi(z) z^{n-\frac{1}{2}}, \tag{3.33}$$

which becomes

$$\psi_n^{(1)} = \int_{C_2} \frac{\mathrm{d}z}{2\pi i} \psi(z) z^{n-\frac{1}{2}} + \int_{C_3} \frac{\mathrm{d}z}{2\pi i} \psi(z) z^{n-\frac{1}{2}} \tag{3.34}$$

The C_2 term is what we have computed for the fermions previously, and is given by

$$\int_{C_2} \frac{\mathrm{d}z}{2\pi i} \psi(z) z^{n-\frac{1}{2}} = \frac{1}{2} \sum_k \psi_k \int_{C_2} \frac{\mathrm{d}z}{2\pi i} z^{n-\frac{k}{2}-1}$$

$$= \frac{1}{2} \psi_{2n} \pm \frac{i}{2\pi} \sum_{k \, \mathrm{odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}} \psi_k. \tag{3.35}$$

The C_3 contribution also goes to zero for the fermions. When one goes to the t-plane, one finds

$$\int_{\tilde{C}_3} \frac{\mathrm{d}t}{2\pi i} \psi(t) \sqrt{2t} (z_0 + t^2)^{n-\frac{1}{2}}, \tag{3.36}$$

which acts on the spin field $S(t = 0)$. The most singular term in the OPE between $\psi(t)$ and $S(0)$ is proportional to $1/\sqrt{t}$, and so one again finds that the semi-circular contour vanishes.

At this point, we see the resolution of the first problem we found with our formal derivation. If the field has an OPE with the spin-field in the covering space which is singular enough, then the C_3-contribution is nonvanishing in the limit as $\tilde{\varepsilon} \to 0$. This extra contribution gives exactly the correct answer for J_0^-, for instance, as we demonstrate in Section 4.1.

3.3.3. Multiple contours

Having resolved the first problem with the formal derivation, we should now discuss the UV issues that arise with multiple contours.[5]

Let us write ε_i for the radius of the ith mode's C_3 semi-circle. For instance, consider an initial state

$$\alpha_{n_1}^{(1)} \alpha_{n_2}^{(1)} |0_R^-\rangle^{(1)} |0_R^-\rangle^{(2)}. \tag{3.37}$$

Then the C_3 part coming from $\alpha_{n_1}^{(1)}$ has radius ε_1 in the z-plane and therefore the semi-circle in the t-plane has radius $\sqrt{\varepsilon_1}$. Similarly, for $\alpha_{n_2}^{(1)}$. If we require that the C_2 parts of α_{n_1} and α_{n_2} preserve the same ordering, then the semi-circles in the t-plane satisfy

$$\varepsilon_2 < \varepsilon_1. \tag{3.38}$$

---

[5] Recall that the definition of the twist operator σ_2 involves a hole in the z-plane, whose radius is carefully taken to zero [36]. One might suspect that this limit has important consequences for these UV issues and that the size of the hole plays the role of a UV cutoff. In fact, the hole is taken to zero size *before* any of the issues discussed in this paper, and does not play any role here. The actual issue is the interaction between neighboring contours, as discussed.

We now argue that, in fact, we should have $\varepsilon_2 \ll \varepsilon_1$ in order to use the intertwining relations as we want to. Consider the two semi-circular C_3 contours at leading order in ε_1 and ε_2:

$$
\int_{\tilde{C}_3(\varepsilon_1)} \frac{dt}{2\pi i} \partial X(t)(z_0 + t^2)^{n_1} \int_{\tilde{C}_3(\varepsilon_2)} \frac{dt'}{2\pi i} \partial X(t')(z_0 + t'^2)^{n_2} \sim \int_{\tilde{C}_3(\varepsilon_1)} \frac{dt}{2\pi i}(z_0 + t^2)^{n_1} \int_{\tilde{C}_3(\varepsilon_2)} \frac{dt'}{2\pi i} \frac{(z_0 + t'^2)^{n_2}}{(t - t')^2}
$$

$$
\sim \int_{\tilde{C}_3(\varepsilon_1)} \frac{dt}{2\pi i} z_0^{n_1} \int_{\tilde{C}_3(\varepsilon_2)} \frac{dt'}{2\pi i} \frac{z_0^{n_2}}{(t - t')^2}
$$

$$
\sim z_0^{n_1 + n_2} \log \frac{1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}{1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}
$$

$$
\sim z_0^{n_1 + n_2} \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} + \cdots . \tag{3.39}
$$

This vanishes only if $\varepsilon_2 \ll \varepsilon_1$. Note that this argument is unaffected if the modes are on different copies (the integrals work out in essentially the same way).

Taking $\varepsilon_2 \ll \varepsilon_1$ suggests a particular way to take the limit for double-sums: if L_i is the cutoff on the $\alpha_{n_i}^{(1)}$-sum, then we should take

$$
L_2 \gg L_1, \tag{3.40}
$$

that is, evaluate the L_2-sum first and then the L_1-sum. This is exactly the way that we got the correct answer, when the issue was demonstrated in (3.22).

3.3.4. The prescription

We now develop the precise prescription that resolves the UV ambiguities. Before we state the prescription, we should mention that there are two kinds of sums over modes. There are the after-the-twist intertwining sums, which have ordering ambiguities among themselves, and there can also be before-the-twist sums on modes before the twist operator. For example, consider a composite operator like

$$
J_n^{a(1)} = -\frac{1}{4}(\sigma^{aT})^{\alpha}{}_{\beta} \sum_{j=-\infty}^{\infty} \psi_{\alpha A, n-j}^{(1)\dagger} \psi_j^{(1)\beta A}. \tag{3.41}
$$

In fact, these sums also have UV-limit issues when combined with the intertwining relations as is demonstrated in Section 4.2. If we look at the intertwining relation in (3.8), for example, with the implicit cutoff on the sum,

$$
\alpha_m^{(2)} = \frac{1}{2}\alpha_{2m} - \frac{i}{2\pi} \sum_{\substack{n \text{ odd}}}^{|n| < L} \frac{z_0^{m - \frac{n}{2}}}{m - \frac{n}{2}} \alpha_n, \tag{3.42}
$$

we see that we are approximating a mode m as a linear combination of modes with UV cutoff L. In order for this approximation to become an exact expansion we must take $L \to \infty$ *with m fixed.* That is we need to have much higher frequency modes in our sum than the mode that we are expanding. Therefore, we need to cutoff the before-the-twist sum in (3.41) and ensure that its cutoff is much less than the after-the-twist cutoff in (3.42).

Before proceeding, let us consider more generally what sort of multi-dimensional series we can get in this formalism. If we have a bunch of modes in the initial state that we pull across using

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the intertwining relations, then there are several different types of terms that can arise. There is a term in which all of the positive modes act on $|\chi\rangle$ separately, and we are left with a product of one-dimensional sums that result in either (3.16) or (3.18). Then, there are terms where the positive modes from one sum contract with negative modes of another. This gives a double sum, like the one in (3.23). If there is a sum on the before-the-twist modes, then we can get a triple sum if those two modes contract. This is the most complicated sum possible.

Finally, we are ready to state the prescription that ensures that the multi-dimensional series converge to the correct answer. The prescription is

1. The after-the-twist intertwining sums should be performed from innermost contour (right-most mode) to outermost contour (left-most mode). This ensures that the C_3-terms can be dropped.

2. Any sums on before-the-twist modes should be performed *last*. There is no UV-ambiguity among the before-the-twist sums.

Note that with this prescription we have a weakened version of our goal. Because of the UV sensitive series, one cannot directly map operators to operators since one requires knowledge of what other modes are around in order to correctly evaluate the series. While the modes that we start with before the twist operator may commute, we need to think about them pulling across the twist operator in a particular order. It is in this sense, that we have intertwining relations and not Bogolyubov coefficients.

4. AN EXAMPLE: INTERTWINING RELATIONS FOR J_n^a

There are two equivalent ways of calculating the effect of the twist operator on composite operators such as J_n^a. One way is to use the contour deformation method described in this paper, being careful not to throw away the small contour C_3. The other way is to write J^a as the product of fermion modes and use the intertwining relations for the fermions, being careful to use the prescriptions described in Section 3.3.4.

4.1. The contour method

We first describe the contour method mentioned above. If we go through the argument described in Section 3.3 then

$$J_n^{a(1)} = \frac{1}{2} J_{2n}^a + \frac{i}{2\pi} \sum_{k \, \text{odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}} J_k^a + \lim_{\varepsilon \to 0} \int_{C_3} \frac{\mathrm{d}z}{2\pi i} J^{a(1)}(z) z^n, \tag{4.1}$$

but this time we find a nonvanishing contribution coming from the C_3 contour. As we pull the $J_n^{a(1)}$ contour out, it acts on the twist operator (or equivalently, the spin field in the covering space) and can switch a σ_2^+ to σ_2^-. Therefore, let us consider σ_2^α.

We can evaluate the C_3 contour by going to the t-plane, where $\sigma_2^\alpha(z_0)$ leaves only a spin field $S^\alpha(t = 0)$. The image of C_3 in the t-plane, $\tilde{C}_3$, is a semi-circle around the origin as shown in

Figure 3. Thus, the $\tilde{C}_3$ contour implicitly acts on the spin field:

$$
\begin{aligned}
\int_{C_3} \frac{dz}{2\pi i} J^{(1)a}(z) z^n &= \left[\int_{\tilde{C}_3} \frac{dt}{2\pi i} J^a(t)(z_0 + t^2)^n \right] S^\alpha(0) \\
&= \frac{1}{2} (\sigma^{aT})^\alpha{}_\beta S^\beta(0) \int_{\tilde{C}_3} \frac{dt}{2\pi i} \frac{(z_0 + t^2)^n}{t} \\
&= \frac{1}{2} (\sigma^{aT})^\alpha{}_\beta S^\beta(0) \frac{1}{2\pi i} \left(z_0^n \log t \Big|_{-\varepsilon}^{\varepsilon} + O(\varepsilon) \right) \\
&= -\frac{z_0^n}{4} (\sigma^{aT})^\alpha{}_\beta S^\beta(0),
\end{aligned}
\tag{4.2}
$$

where we have given the result after taking $\varepsilon \to 0$. When we go back to the z-plane we should write the above result as

$$
\sigma_2^\alpha(z_0) J_n^{a(1)} = \left[\frac{1}{2} J_{2n}^a + \frac{i}{2\pi} \sum_{k \, \text{odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}} J_k^a \right] \sigma_2^\alpha(z_0) - \frac{z_0^n}{4} (\sigma^{aT})^\alpha{}_\beta \sigma_2^\beta(z_0).
\tag{4.3}
$$

If we had considered $J_n^{a(2)}$, then we obtain a similar result. We can summarize the two relations and write them in a suggestive form as

$$
\sigma_2^\alpha(z_0) J_n^{a(1,2)} = \left[\frac{1}{2} J_{2n}^a \pm \frac{i}{2\pi} \sum_{k \, \text{odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}} J_k^a \right] \sigma_2^\alpha(z_0) - \frac{z_0^n}{2} [J_0^a, \sigma_2^\alpha(z_0)].
\tag{4.4}
$$

We see that if we use these intertwining relations, then we get the correct answer in (3.20).

While the above intertwining relation is correct, it may not be the most useful form. Because we have switched the charge on the twist operator, we now must think about a new state $|\chi\rangle$ created by the negatively-charged operator acting on the vacuum.

4.1.1. An example

For concreteness, let us consider

$$
\sigma_2^+(z_0) J_n^{-(1)} |0^-\rangle^{(1)} |0^-\rangle^{(2)}, \qquad n < 0,
\tag{4.5}
$$

then we have

$$
\left[\frac{1}{2} J_{2n}^- + \frac{i}{2\pi} \sum_{k \, \text{odd}} \frac{z_0^{n-\frac{k}{2}}}{n - \frac{k}{2}} J_k^- - \frac{z_0^n}{2} J_0^- \right] |\chi\rangle.
\tag{4.6}
$$

Thus, our first task is to compute the three terms,

$$
J_{2n}^- |\chi\rangle \qquad J_k^- |\chi\rangle \, k \, \text{odd} \qquad J_0^- |\chi\rangle.
\tag{4.7}
$$

There are no real complications in working the terms out. For instance, for the first term, one starts with[6]

$$
J_{2n}^- = -\frac{1}{4} \sum_j \psi_{+A,2n-j}^\dagger \psi_j^{-A},
\tag{4.8}
$$

---

[6] Note that the J after the twist has an extra factor of $1/2$ from before the twist, which arises from the fermion–fermion anticommutator having an extra factor of 2 after the twist.

then breaks the sum into terms with both modes negative and terms with odd positive modes that act on $|\chi\rangle$. One can write the result in the form

$$J_{2n}^-|\chi\rangle = -\frac{1}{4}\sum_{2n+1\leq j\leq -1}\psi_{+A,2n-j}^\dagger\psi_j^{-A}|\chi\rangle + \sum_{j,p\,\text{odd}^+}\gamma_{jp}^F\psi_{+A,-p}^\dagger\psi_{2n-j}^{-A}|\chi\rangle. \tag{4.9}$$

Similarly, J_0^- becomes

$$J_0^-|\chi\rangle = \sum_{j,p\,\text{odd}^+}\gamma_{jp}^F\psi_{+A,-j}^\dagger\psi_{-p}^{-A}|\chi\rangle. \tag{4.10}$$

The J_k^- term is not much work, but there are two distinct cases, corresponding to whether or not there are terms where both ψs are raising operators: $k\leq -3$ and $k\geq -1$. One finds

$$J_k^-|\chi\rangle = -\frac{1}{4}\sum_{k+1\leq j\leq -1}\psi_{+A,k-j}^\dagger\psi_j^{-A}|\chi\rangle + \sum_{j,p\,\text{odd}^+}\gamma_{jp}^F\psi_{+A,k-j}^\dagger\psi_{-p}^{-A}|\chi\rangle \qquad k\leq -3,\text{odd} \tag{4.11}$$

and

$$J_k^-|\chi\rangle = \sum_{j,p\,\text{odd}^+}\gamma_{k+j+1,p}^F\psi_{+A,-j-1}^\dagger\psi_{-p}^{-A}|\chi\rangle \qquad k\geq -1,\text{odd}. \tag{4.12}$$

The slightly more difficult task is the sum over k, which can be written as

$$\sum_{k\,\text{odd}}^{n-\frac{k}{2}}\frac{z_0}{n-\frac{k}{2}}J_k^-|\chi\rangle = \sum_{k\,\text{odd}}^{k\leq -3}\frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}}J_k^-|\chi\rangle + \sum_{k\,\text{odd}}^{k\geq -1}\frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}}J_k^-|\chi\rangle$$

$$= -\frac{1}{2}\sum_{j,p\,\text{odd}^+}\frac{z_0^{n+\frac{j+p+1}{2}}}{n+\frac{j+p+1}{2}}\psi_{+A,-j-1}^\dagger\psi_{-p}^{-A}$$

$$+ \sum_{j\,\text{odd}}^{j\geq 3}\sum_{p\,\text{odd}^+}\left(\sum_{k\,\text{odd}}^{-j\leq k\leq -3}\frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}}\gamma_{k+j+1,p}^F\right)\psi_{+A,-j-1}^\dagger\psi_{-p}^{-A}$$

$$+ \sum_{j,p\,\text{odd}^+}\left(\sum_{k\,\text{odd}}^{k\geq -1}\frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}}\gamma_{k+j+1,p}^F\right)\psi_{+A,-j-1}^\dagger\psi_{-p}^{-A}, \tag{4.13}$$

after a few manipulations. Note that all of the terms on the right implicitly act on $|\chi\rangle$. Finally, using (B.3b) one arrives at

$$\sum_{k\,\text{odd}}^{n-\frac{k}{2}}\frac{z_0}{n-\frac{k}{2}}J_k^-|\chi\rangle = -\sum_{j,p\,\text{odd}^+}\frac{z_0^{n+\frac{j+p+1}{2}}}{2(n+\frac{j+p+1}{2})}\frac{\Gamma(\frac{p}{2})\Gamma(-n-\frac{j}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n-\frac{j+1}{2})}\psi_{+A,-j-1}^\dagger\psi_{-p}^{-A}|\chi\rangle \tag{4.14}$$

Putting it all together, we have

$$\sigma_2^+(z_0)J_n^{-(1)}|0^-\rangle^{(1)}|0^-\rangle^{(2)} = -\frac{1}{8}\sum_{j=n+1}^{-1}\psi_{+A,2n-2j}^\dagger\psi_{2j}^{-A}|\chi\rangle$$

$$-\frac{i}{4\pi}\sum_{j=1}^{\infty}\sum_{p\,\text{odd}^+}\frac{z_0^{n+j+\frac{p}{2}}}{n+j+\frac{p}{2}}\frac{\Gamma(\frac{p}{2})\Gamma(-n-j+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n-j)}\psi_{+A,-2j}^\dagger\psi_{-p}^{-A}|\chi\rangle$$

$$+\frac{1}{2}\sum_{j,p\,\text{odd}^+}\left[-\frac{1}{4}\delta_{j+p+2n,0}+\frac{\gamma_{2n+p,j}^F+\gamma_{2n+j,p}^F}{2}-z_0^n\frac{\gamma_{jp}^F+\gamma_{pj}^F}{2}\right]\psi_{+A,-j}^\dagger\psi_{-p}^{-A}|\chi\rangle, \tag{4.15}$$

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where for ease of comparison we have broken the result into even–even, even–odd, and odd–odd terms. In the above expression, we define γ^F with negative indices to be zero. Furthermore, we have explicitly symmetrized over j and p in the last term since

$$\psi^\dagger_{+A,-j}\psi^{-A}_{-p} = \psi^\dagger_{+A,-p}\psi^{-A}_{-j}. \tag{4.16}$$

Below, we compare this result with what one finds when one breaks the $J_n^{-(1)}$ into fermions and uses the fermion intertwining relations.

4.2. The composite method

We now show how to reproduce the above result by using the fermion intertwining relations and our prescription.

We start by writing

$$J_n^{a(1)} = -\frac{1}{4}(\sigma^{aT})^\alpha{}_\beta \sum_{j=-\infty}^{\infty} \psi^{(1)\dagger}_{\alpha A,n-j}\psi^{(1)\beta A}_j, \tag{4.17}$$

and note that the sum on j is a sum over "before-the-twist" modes and therefore should be evaluated last according to the prescription. We then can use our intertwining relations to write this directly as

$$J_n^{a(1)} = -\frac{1}{4}(\sigma^{aT})^\alpha{}_\beta \sum_{j=-\infty}^{\infty} \left[\frac{1}{2}\psi^\dagger_{\alpha A,2n-2j} + \frac{i}{2\pi}\sum_{k\,\text{odd}} \frac{z_0^{n-j-\frac{k}{2}}}{n-j-\frac{k}{2}}\psi^\dagger_{\alpha A,k}\right]\left[\frac{1}{2}\psi^{\beta A}_{2j} + \frac{i}{2\pi}\sum_{l\,\text{odd}} \frac{z_0^{j-\frac{l}{2}}}{j-\frac{l}{2}}\psi^{\beta A}_l\right]. \tag{4.18}$$

At this point, we can say nothing further until we know what $J_n^{a(1)}$ acts on. Of course, when faced with an expression like the above it is rather tempting to evaluate the j-sum using

$$\sum_{j=-\infty}^{\infty} \frac{1}{(n-j-\frac{k}{2})(j-\frac{l}{2})} = -\pi^2\delta_{\frac{l}{2},n-\frac{k}{2}} \qquad k,l\,\text{odd}, \tag{4.19}$$

which immediately leads to the false relation

$$J_n^{a(1)} = \frac{1}{2}J_{2n}^a + \frac{i}{2\pi}\sum_{k\,\text{odd}} \frac{z_0^{n-\frac{k}{2}}}{n-\frac{k}{2}}J_k^a \qquad \text{(false relation)}. \tag{4.20}$$

This demonstrates the need for the restriction on before-the-twist sums.

In order to proceed and compare to (4.4), let us again consider the state

$$\sigma_2^+(z_0)J_n^{-(1)}|0_R^-\rangle^{(1)}|0_R^-\rangle^{(2)}. \tag{4.21}$$

Now, one could make the j-sum in (4.18) finite by considering the above; however, our prescription ensures that one gets the correct answer even if one leaves it as an infinite series.

If we act on the vacuum as in (4.21), then we get (4.18) acting on $|\chi\rangle$. The $\psi^{\beta A}$ is the rightmost mode and so the l-sum should be evaluated first, followed by the k-sum, and then finally the j-sum. We can use (3.18) to quickly read off the result of the l-sum. There are now two distinct terms to consider for the k-sum. There is the possibility of the $\psi^\dagger_{\alpha A}$ contracting with the result of the l-sum, and the $\psi^\dagger$ can pass through and act on $|\chi\rangle$ (and we can again use (3.18)). The contraction gives zero. For other composite operators, however, the contraction term can be nonzero.

Following the above procedure, we get

$$
-\frac{1}{2}\sum_{j=-\infty}^{\infty}\left[\frac{1}{2}\psi^\dagger_{+A,2(n-j)}+\frac{i}{2\pi}\sum_{p\,\mathrm{odd}^+}\frac{z_0^{\,n-j+\frac{p}{2}}}{n-j+\frac{p}{2}}\frac{\Gamma(\frac{p}{2})\Gamma(-n+j+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n+j)}\psi^\dagger_{+A,-p}\right]
$$
$$
\left[\frac{1}{2}\psi^{-A}_{2j}+\frac{i}{2\pi}\sum_{q\,\mathrm{odd}^+}\frac{z_0^{\,j+\frac{q}{2}}}{j+\frac{q}{2}}\frac{\Gamma(\frac{q}{2})\Gamma(-j+\frac{1}{2})}{\Gamma(\frac{q+1}{2})\Gamma(-j)}\psi^{-A}_{-q}\right].
\tag{4.22}
$$

There are four terms from the above multiplication.

The even–even term is

$$
(\text{even–even})=-\frac{1}{8}\sum_{j=n+1}^{-1}\psi^\dagger_{+A,2n-2j}\psi^{-A}_{2j}.
\tag{4.23}
$$

One can show that the two even–odd cross-terms are identical, and sum to

$$
(\text{even–odd})=-\frac{i}{4\pi}\sum_{j=-\infty}^{-1}\sum_{p\,\mathrm{odd}^+}\frac{z_0^{\,n-j+\frac{p}{2}}}{n-j+\frac{p}{2}}\frac{\Gamma(\frac{p}{2})\Gamma(-n+j+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n+j)}\psi^\dagger_{+A,-p}\psi^{-A}_{2j}
\tag{4.24}
$$

One may write the odd–odd term as

$$
(\text{odd–odd})=\frac{1}{8\pi^2}\sum_{p,q\,\mathrm{odd}^+}\psi^\dagger_{+A,-p}\psi^{-A}_{-q}z_0^{\,n+\frac{p+q}{2}}\frac{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}S(n,p,q),
\tag{4.25}
$$

where

$$
S(n,p,q)=\sum_{j=n+1}^{-1}\frac{1}{(n-j+\frac{p}{2})(j+\frac{q}{2})}\frac{\Gamma(-n+j+\frac{1}{2})\Gamma(-j+\frac{1}{2})}{\Gamma(-n+j)\Gamma(-j)}.
\tag{4.26}
$$

Comparing the above to the three terms in Equation (4.15), one finds agreement provided

$$
\frac{z_0^{\,n+\frac{j+p}{2}}}{8\pi^2}\frac{\Gamma(\frac{j}{2})\Gamma(\frac{p}{2})}{\Gamma(\frac{j+1}{2})\Gamma(\frac{p+1}{2})}S(n,j,p)=\frac{1}{2}\left[-\frac{1}{4}\delta_{j+p+2n,0}+\frac{\gamma^F_{2n+p,j}+\gamma^F_{2n+j,p}}{2}-z_0^{\,n}\frac{\gamma^F_{jp}+\gamma^F_{pj}}{2}\right].
\tag{4.27}
$$

This equation follows from the identity

$$
\sum_{k=0}^{\mu}\frac{1}{(\mu-k-\frac{\alpha}{2})(k-\frac{\beta}{2})}\frac{\Gamma(\mu-k+\frac{3}{2})\Gamma(k+\frac{3}{2})}{\Gamma(\mu-k+1)\Gamma(k+1)}
$$
$$
=\pi-\frac{\pi}{\frac{\alpha+\beta}{2}-\mu}\left(\frac{\Gamma(\frac{\alpha+3}{2})\Gamma(\frac{\alpha}{2}-\mu)}{\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha-1}{2}-\mu)}+\frac{\Gamma(\frac{\beta+3}{2})\Gamma(\frac{\beta}{2}-\mu)}{\Gamma(\frac{\beta}{2}+1)\Gamma(\frac{\beta-1}{2}-\mu)}\right),
\tag{4.28}
$$

where the left-hand side is $S(n,p,q)$ with $\mu=-n-2$, $\alpha=p-2$, and $\beta=q-2$. If we call the above sum $F(\mu,\alpha,\beta)$, then one can prove the identity by showing that both sides of Equation (4.28) obey the four-term recursion relation

$$
f_0(\mu,\alpha,\beta)F(\mu,\alpha,\beta)+f_1(\mu,\alpha,\beta)F(\mu+1,\alpha,\beta)
$$
$$
+f_2(\mu,\alpha,\beta)F(\mu+2,\alpha,\beta)+f_3(\mu,\alpha,\beta)F(\mu+3,\alpha,\beta)=0,
\tag{4.29}
$$

with

$$f_0(\mu, \alpha, \beta) = (-3 + \alpha - 2\mu)(\alpha + \beta - 2\mu)(3 - \beta + 2\mu)(\alpha + \beta - 6(3 + \mu)) \tag{4.30a}$$

$$\begin{aligned} f_1(\mu, \alpha, \beta) &= 3(\alpha + \beta - 2(1 + \mu))\big(- 2\beta^2(2 + \mu) + \alpha^2(\beta - 2(2 + \mu)) + \beta(79 + 8\mu(9 + 2\mu)) \\ &\quad + \alpha \left(79 + \beta^2 + 8\mu(9 + 2\mu) - 2\beta(12 + 5\mu)\right) - 2(128 + \mu(175 + 4\mu(20 + 3\mu)))\big) \end{aligned} \tag{4.30b}$$

$$\begin{aligned} f_2(\mu, \alpha, \beta) &= 3(\alpha + \beta - 2(2 + \mu))\big(\beta^2(5 + 2\mu) + \alpha^2(5 - \beta + 2\mu) + 4(2 + \mu)\left(43 + 32\mu + 6\mu^2\right) \\ &\quad - 2\beta(47 + \mu(39 + 8\mu)) - \alpha\left(94 + \beta^2 - 2\beta(12 + 5\mu) + 2\mu(39 + 8\mu)\right)\big) \end{aligned} \tag{4.30c}$$

$$f_3(\mu, \alpha, \beta) = (\alpha + \beta - 6(2 + \mu))(\alpha - 2(3 + \mu))(\beta - 2(3 + \mu))(\alpha + \beta - 2(3 + \mu)). \tag{4.30d}$$

Thus the identity holds by induction once one confirms that it holds for $\mu = 0, 1, 2$. It may be helpful to use an algebraic manipulation program such as *Mathematica* to show the above.

5. CONCLUSION

We presented a method for computing the effect of the blow-up mode deformation operator on untwisted states of the D1D5 CFT. An alternative method was developed in [35]; however, the method developed here comes in the form of intertwining relations for the 2-twist operator, $\sigma_2^+(z_0)$. This method, therefore, is more direct and of theoretical interest since it makes the physics more transparent. We would be remiss, if we did not note that both the method in [35] and that presented here should generalize to higher order twist operators.

Using the technology and understanding developed in [34, 35] and here, we hope to address some important outstanding questions concerning the D1D5 CFT and black hole physics. In particular, we hope to elucidate the nature of the proposed "non-renormalization" theorem [33]; how states of the CFT fragment, thereby "scrambling" information [42, 43]; black hole formation; and the in-falling observer. All of these are important and deep issues that will probably require lots of work to resolve; however, this paper should serve as a step toward these goals. All of the discussion here and in [34, 35] has been off-shell. A more modest next step would be to analyze the physics that arise on-shell. Another task is to systematically analyze the combinatorial factors that arise at different orders in perturbation theory, perhaps using the technology developed in [44, 45].

Appendix A: Notation and the CFT algebra

For reference, we give the notation and conventions we use (see also [34, 35]). What we describe below is the field content and algebra of a single copy of the CFT in the untwisted sector.

We have 4 real left-moving fermions, ψ_1 through ψ_4, which we group into doublets $\psi^{\alpha A}$:

$$\begin{pmatrix} \psi^{++} \\ \psi^{-+} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_3 + i\psi_4 \end{pmatrix} \tag{A.1a}$$

$$\begin{pmatrix} \psi^{+-} \\ \psi^{--} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_3 - i\psi_4 \\ -(\psi_1 - i\psi_2) \end{pmatrix} \tag{A.1b}$$

Here $\alpha = (+,-)$ is an index of the subgroup $SU(2)_L$ of rotations on S^3 and $A = (+,-)$ is an index of the subgroup $SU(2)_1$ from rotations in T^4. The 2-point functions are

$$\langle \psi^{\alpha A}(z)\psi^{\beta B}(w) \rangle = -\epsilon^{\alpha\beta}\epsilon^{AB} \frac{1}{z-w}, \tag{A.2}$$

where we have defined the ϵ symbol as

$$\epsilon_{+-} = 1 \qquad \epsilon^{+-} = -1. \tag{A.3}$$

Hermitian conjugation is defined as

$$\psi^{\dagger}_{\alpha A} = -\epsilon_{\alpha\beta}\epsilon_{AB}\psi^{\beta B}. \tag{A.4}$$

There are 4 real left-moving bosons, X_1 through X_4, which can be grouped into a matrix

$$X_{A\dot{A}} = \frac{1}{\sqrt{2}} X_i \sigma_i = \frac{1}{\sqrt{2}} \begin{pmatrix} X_3 + iX_4 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 + iX_4 \end{pmatrix}, \tag{A.5}$$

where σ_i with $i = 1, 2, 3$ are the Pauli matrices and $\sigma_4 = iI$. The 2-point functions are

$$\langle \partial X_{A\dot{A}}(z)\partial X_{B\dot{B}}(w) \rangle = \frac{1}{(z-w)^2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}. \tag{A.6}$$

The bosonic and fermionic modes satisfy

$$\left[\alpha_{m,A\dot{A}}, \alpha_{n,B\dot{B}} \right] = m\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\delta_{m+n,0}, \tag{A.7}$$

and

$$\{\psi_m^{\alpha A}, \psi_n^{\beta B}\} = -\epsilon^{\alpha\beta}\epsilon^{AB}\delta_{m+n,0}, \tag{A.8}$$

respectively.

The chiral algebra is generated by the operators

$$J^a = -\frac{1}{4}(\psi^{\dagger})_{\alpha A}(\sigma^{Ta})^{\alpha}_{\ \beta}\psi^{\beta A} \tag{A.9a}$$

$$G_{\dot{A}}^{\alpha} = \psi^{\alpha A}\partial X_{A\dot{A}}, \quad (G^{\dagger})_{\alpha}^{\dot{A}} = (\psi^{\dagger})_{\alpha A}\partial(X^{\dagger})^{A\dot{A}} \tag{A.9b}$$

$$T = -\frac{1}{2}(\partial X^{\dagger})^{A\dot{A}}\partial X_{A\dot{A}} - \frac{1}{2}(\psi^{\dagger})_{\alpha A}\partial\psi^{\alpha A} \tag{A.9c}$$

$$(G^{\dagger})_{\alpha}^{\dot{A}} = -\epsilon_{\alpha\beta}\epsilon^{\dot{A}\dot{B}}G_{\dot{B}}^{\beta}, \quad G_{\dot{A}}^{\alpha} = -\epsilon^{\alpha\beta}\epsilon_{\dot{A}\dot{B}}(G^{\dagger})_{\beta}^{\dot{B}}. \tag{A.9d}$$

These operators generate the algebra

$$J^a(z)J^b(w) \sim \delta^{ab}\frac{\frac{1}{2}}{(z-w)^2} + i\epsilon^{abc}\frac{J^c}{z-w} \tag{A.10a}$$

$$J^a(z)G^\alpha_{\dot{A}}(z') \sim \frac{1}{(z-z')}\frac{1}{2}(\sigma^{aT})^\alpha{}_\beta G^\beta_{\dot{A}} \tag{A.10b}$$

$$G^\alpha_{\dot{A}}(z)(G^\dagger)^{\dot{B}}_\beta(z') \sim -\frac{2}{(z-z')^3}\delta^\alpha_\beta\delta^{\dot{B}}_{\dot{A}} - \delta^{\dot{B}}_{\dot{A}}(\sigma^{Ta})^\alpha{}_\beta\left[\frac{2J^a}{(z-z')^2} + \frac{\partial J^a}{(z-z')}\right] - \frac{1}{(z-w)}\delta^\alpha_\beta\delta^{\dot{B}}_{\dot{A}}T \tag{A.10c}$$

$$T(z)T(z') \sim \frac{3}{(z-z')^4} + \frac{2T}{(z-z')^2} + \frac{\partial T}{(z-z')} \tag{A.10d}$$

$$T(z)J^a(z') \sim \frac{J^a}{(z-z')^2} + \frac{\partial J^a}{(z-z')} \tag{A.10e}$$

$$T(z)G^\alpha_{\dot{A}}(z') \sim \frac{\frac{3}{2}G^\alpha_{\dot{A}}}{(z-z')^2} + \frac{\partial G^\alpha_{\dot{A}}}{(z-z')}. \tag{A.10f}$$

Note that

$$J^a(z)\psi^{\gamma C}(w) \sim \frac{1}{2}\frac{1}{z-w}(\sigma^{aT})^\gamma{}_\beta\psi^{\beta C}. \tag{A.11}$$

The above OPE algebra gives the commutation relations

$$\left[J^a_m, J^b_n\right] = \frac{m}{2}\delta^{ab}\delta_{m+n,0} + i\epsilon^{ab}{}_c J^c_{m+n} \tag{A.12a}$$

$$\left[J^a_m, G^\alpha_{\dot{A},n}\right] = \frac{1}{2}(\sigma^{aT})^\alpha{}_\beta G^\beta_{\dot{A},m+n} \tag{A.12b}$$

$$\{G^\alpha_{\dot{A},m}, G^\beta_{\dot{B},n}\} = \epsilon_{\dot{A}\dot{B}}\left[(m^2 - \frac{1}{4})\epsilon^{\alpha\beta}\delta_{m+n,0} + (m-n)(\sigma^{aT})^\alpha{}_\gamma\epsilon^{\gamma\beta}J^a_{m+n} + \epsilon^{\alpha\beta}L_{m+n}\right] \tag{A.12c}$$

$$[L_m, L_n] = \frac{m(m^2 - \frac{1}{4})}{2}\delta_{m+n,0} + (m-n)L_{m+n} \tag{A.12d}$$

$$[L_m, J^a_n] = -nJ^a_{m+n} \tag{A.12e}$$

$$\left[L_m, G^\alpha_{\dot{A},n}\right] = \left(\frac{m}{2} - n\right)G^\alpha_{\dot{A},m+n}. \tag{A.12f}$$

Appendix B: Useful Series

We collect the series that we use throughout the paper here. These series arise when considering a single boson and fermion.

$$\sum_{k\,\text{odd}^+}\frac{1}{(n-\frac{k}{2})(k+l)}\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})} = \frac{\pi l\Gamma(\frac{l+1}{2})}{4\Gamma(\frac{l}{2}+1)}\frac{1}{n+\frac{l}{2}}\left(\frac{\Gamma(\frac{l}{2})\Gamma(-n+\frac{1}{2})}{\Gamma(\frac{l+1}{2})\Gamma(-n)} - 1\right) \tag{B.1}$$

$$\sum_{k\,\text{odd}^+}\frac{1}{k(p+k)(n-\frac{k}{2})}\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})} = -\frac{\pi\Gamma(\frac{p+1}{2})}{4\Gamma(\frac{p}{2}+1)}\frac{1}{n+\frac{p}{2}}\left(\frac{\Gamma(\frac{p}{2}+1)\Gamma(-n+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(1-n)} - 1\right) \tag{B.2}$$

These series arise when considering a single fermion:

$$\sum_{k\,\text{odd}^+} \frac{z_0^{n-\frac{k}{2}}\gamma_{pk}^F}{n-\frac{k}{2}} = \frac{z_0^{n+\frac{p}{2}}}{2(n+\frac{p}{2})}\left(\frac{\Gamma(\frac{p}{2}+1)\Gamma(-n+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n+1)}-1\right) \tag{B.3a}$$

$$\sum_{k\,\text{odd}^+} \frac{z_0^{n-\frac{k}{2}}\gamma_{kp}^F}{n-\frac{k}{2}} = -\frac{z_0^{n+\frac{p}{2}}}{2(n+\frac{p}{2})}\left(\frac{\Gamma(\frac{p}{2})\Gamma(-n+\frac{1}{2})}{\Gamma(\frac{p+1}{2})\Gamma(-n)}-1\right). \tag{B.3b}$$

Some other useful series are

$$\sum_{k\,\text{odd}^+} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+1}{2})(m+\frac{k}{2})} = \pi\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \tag{B.4a}$$

$$\sum_{l\,\text{odd}^+} \frac{l}{(m-\frac{l}{2})(n-\frac{l}{2})}\frac{\Gamma(\frac{l}{2})}{\Gamma(\frac{l+1}{2})} = 2\pi^2 m\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\delta_{m,n} \qquad m,n>0 \tag{B.4b}$$

[1] A. Sen, "Black hole solutions in heterotic string theory on a torus," *Nucl.Phys.* **B440** (1995) 421–440, `arXiv:hep-th/9411187 [hep-th]`.

[2] A. Sen, "Extremal black holes and elementary string states," *Mod.Phys.Lett.* **A10** (1995) 2081–2094, `arXiv:hep-th/9504147 [hep-th]`.

[3] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," *Phys. Lett.* **B379** (1996) 99–104, `arXiv:hep-th/9601029`.

[4] C. G. Callan and J. M. Maldacena, "D-brane Approach to Black Hole Quantum Mechanics," *Nucl. Phys.* **B472** (1996) 591–610, `arXiv:hep-th/9602043`.

[5] A. Dhar, G. Mandal, and S. R. Wadia, "Absorption vs decay of black holes in string theory and T-symmetry," *Phys. Lett.* **B388** (1996) 51–59, `arXiv:hep-th/9605234`.

[6] S. R. Das and S. D. Mathur, "Comparing decay rates for black holes and D-branes," *Nucl. Phys.* **B478** (1996) 561–576, `arXiv:hep-th/9606185`.

[7] S. R. Das and S. D. Mathur, "Interactions involving D-branes," *Nucl. Phys.* **B482** (1996) 153–172, `arXiv:hep-th/9607149`.

[8] J. M. Maldacena and A. Strominger, "Black hole greybody factors and D-brane spectroscopy," *Phys. Rev.* **D55** (1997) 861–870, `arXiv:hep-th/9609026`.

[9] J. Callan, Curtis G., S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, "Absorption of fixed scalars and the D-brane approach to black holes," *Nucl. Phys.* **B489** (1997) 65–94, `arXiv:hep-th/9610172`.

[10] B. D. Chowdhury and A. Virmani, "Modave Lectures on Fuzzballs and Emission from the D1-D5 System," `arXiv:1001.1444 [hep-th]`.

[11] O. J. C. Dias, R. Emparan, and A. Maccarrone, "Microscopic Theory of Black Hole Superradiance," *Phys. Rev.* **D77** (2008) 064018, `arXiv:0712.0791 [hep-th]`.

[12] B. D. Chowdhury and S. D. Mathur, "Radiation from the non-extremal fuzzball," *Class. Quant. Grav.* **25** (2008) 135005, `arXiv:0711.4817 [hep-th]`.

[13] B. D. Chowdhury and S. D. Mathur, "Pair creation in non-extremal fuzzball geometries," *Class. Quant. Grav.* **25** (2008) 225021, `arXiv:0806.2309 [hep-th]`.

[14] B. D. Chowdhury and S. D. Mathur, "Non-extremal fuzzballs and ergoregion emission," *Class. Quant. Grav.* **26** (2009) 035006, `arXiv:0810.2951 [hep-th]`.

[15] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Emission from the D1D5 CFT," *JHEP* **10** (2009) 065, `arXiv:0906.2015 [hep-th]`.

[16] S. G. Avery and B. D. Chowdhury, "Emission from the D1D5 CFT: Higher Twists," *JHEP* **01** (2010) 087, `arXiv:0907.1663 [hep-th]`.

[17] S. D. Mathur, "The fuzzball proposal for black holes: An elementary review," *Fortsch. Phys.* **53** (2005) 793–827, `arXiv:hep-th/0502050`.

[18] S. D. Mathur, "The quantum structure of black holes," *Class. Quant. Grav.* **23** (2006) R115, `arXiv:hep-th/0510180`.

[19] I. Bena and N. P. Warner, "Black holes, black rings and their microstates," *Lect. Notes Phys.* **755** (2008) 1–92, `arXiv:hep-th/0701216`.

[20] K. Skenderis and M. Taylor, "The fuzzball proposal for black holes," *Phys. Rept.* **467** (2008) 117–171, `arXiv:0804.0552 [hep-th]`.

[21] V. Balasubramanian, J. de Boer, S. El-Showk, and I. Messamah, "Black Holes as Effective Geometries," *Class. Quant. Grav.* **25** (2008) 214004, `arXiv:0811.0263 [hep-th]`.

[22] I. Bena, N. Bobev, S. Giusto, C. Ruef, and N. P. Warner, "An Infinite-Dimensional Family of Black-Hole Microstate Geometries," `arXiv:1006.3497 [hep-th]`.

[23] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, "A (Running) Bolt for New Reasons," *JHEP* **11** (2009) 089, `arXiv:0909.2559 [hep-th]`.

[24] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, "Supergravity Solutions from Floating Branes," *JHEP* **03** (2010) 047, `arXiv:0910.1860 [hep-th]`.

[25] J. de Boer and M. Shigemori, "Exotic branes and non-geometric backgrounds," *Phys. Rev. Lett.* **104** (2010) 251603, `arXiv:1004.2521 [hep-th]`.

[26] N. Seiberg and E. Witten, "The D1/D5 system and singular CFT," *JHEP* **04** (1999) 017, `arXiv:hep-th/9903224`.

[27] F. Larsen and E. J. Martinec, "U(1) charges and moduli in the D1-D5 system," *JHEP* **06** (1999) 019, `arXiv:hep-th/9905064`.

[28] J. de Boer, "Six-dimensional supergravity on S**3 x AdS(3) and 2d conformal field theory," *Nucl. Phys.* **B548** (1999) 139–166, `arXiv:hep-th/9806104`.

[29] R. Dijkgraaf, "Instanton strings and hyperKaehler geometry," *Nucl. Phys.* **B543** (1999) 545–571, `arXiv:hep-th/9810210`.

[30] G. E. Arutyunov and S. A. Frolov, "Virasoro amplitude from the S(N) R**24 orbifold sigma model," *Theor. Math. Phys.* **114** (1998) 43–66, `arXiv:hep-th/9708129`.

[31] G. E. Arutyunov and S. A. Frolov, "Four graviton scattering amplitude from S(N) R**8 supersymmetric orbifold sigma model," *Nucl. Phys.* **B524** (1998) 159–206, `arXiv:hep-th/9712061`.

[32] A. Jevicki, M. Mihailescu, and S. Ramgoolam, "Gravity from CFT on S**N(X): Symmetries and interactions," *Nucl. Phys.* **B577** (2000) 47–72, `arXiv:hep-th/9907144`.

[33] J. R. David, G. Mandal, and S. R. Wadia, "Microscopic formulation of black holes in string theory," *Phys.Rept.* **369** (2002) 549–686, `arXiv:hep-th/0203048 [hep-th]`.

[34] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Deforming the D1D5 CFT away from the orbifold point," *JHEP* **06** (2010) 031, `arXiv:1002.3132 [hep-th]`.

[35] S. G. Avery, B. D. Chowdhury, and S. D. Mathur, "Excitations in the deformed D1D5 CFT," *JHEP* **06** (2010) 032, `arXiv:1003.2746 [hep-th]`.

[36] O. Lunin and S. D. Mathur, "Correlation functions for M(N)/S(N) orbifolds," *Commun. Math. Phys.* **219** (2001) 399–442, `arXiv:hep-th/0006196`.

[37] O. Lunin and S. D. Mathur, "Three-point functions for M(N)/S(N) orbifolds with N = 4 supersymmetry," *Commun. Math. Phys.* **227** (2002) 385–419, `arXiv:hep-th/0103169`.

[38] J. R. David, G. Mandal, and S. R. Wadia, "D1/D5 moduli in SCFT and gauge theory, and Hawking radiation," *Nucl. Phys.* **B564** (2000) 103–127, `arXiv:hep-th/9907075`.

[39] E. Gava and K. S. Narain, "Proving the pp-wave / CFT(2) duality," *JHEP* **12** (2002) 023, `arXiv:hep-th/0208081`.

[40] J. Gomis, L. Motl, and A. Strominger, "pp-wave / CFT(2) duality," *JHEP* **11** (2002) 016, `arXiv:hep-th/0206166`.

[41] P.-G. de Gennes, *Superconductivity in Metals and Alloys.* Benjamin, New York, 1966.

[42] Y. Sekino and L. Susskind, "Fast Scramblers," *JHEP* **10** (2008) 065, `arXiv:0808.2096 [hep-th]`.

[43] P. Hayden and J. Preskill, "Black holes as mirrors: quantum information in random subsystems," *JHEP* **09** (2007) 120, `arXiv:0708.4025 [hep-th]`.

[44] A. Pakman, L. Rastelli, and S. S. Razamat, "Extremal Correlators and Hurwitz Numbers in Symmetric Product Orbifolds," `arXiv:0905.3451 [hep-th]`.

[45] A. Pakman, L. Rastelli, and S. S. Razamat, "Diagrams for Symmetric Product Orbifolds," `arXiv:0905.3448 [hep-th]`.